Bifurcations of Dynamical Systems and Numerics
University of Zagreb

The Einstein Relation on Metric Measure Spaces

Uta Freiberg
Technische Universität Chemnitz

11. Mai 2023

Motivation: Analysis and stochastics on fractals

Plan of the lecture

1. (Self–similar) fractals

2. Einstein's relation (on open sets)

3. Einstein's relation on the Sierpinski gasket
Hausdorff, spectral and walk dimension of the SG

4. Upshot, further examples and non–examples

5. ER on MMSs

1. Introduction: Self similar fractals

1.1. Definition and Examples

$K \subseteq \mathbb{R}^n$ is called self similar, if

$$K = \bigcup_{i=1}^{M} S_i(K)$$

where $M \geq 2$ and $S_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ similitudes.

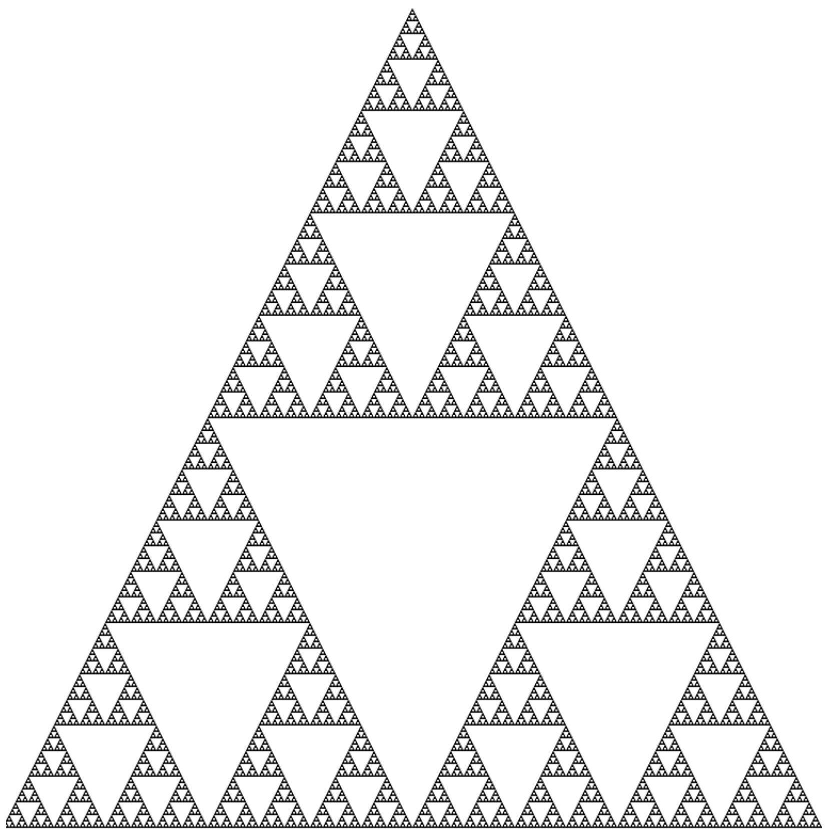
Exp. Sierpinski gasket:

A, B, C vertices of a unilateral triangel

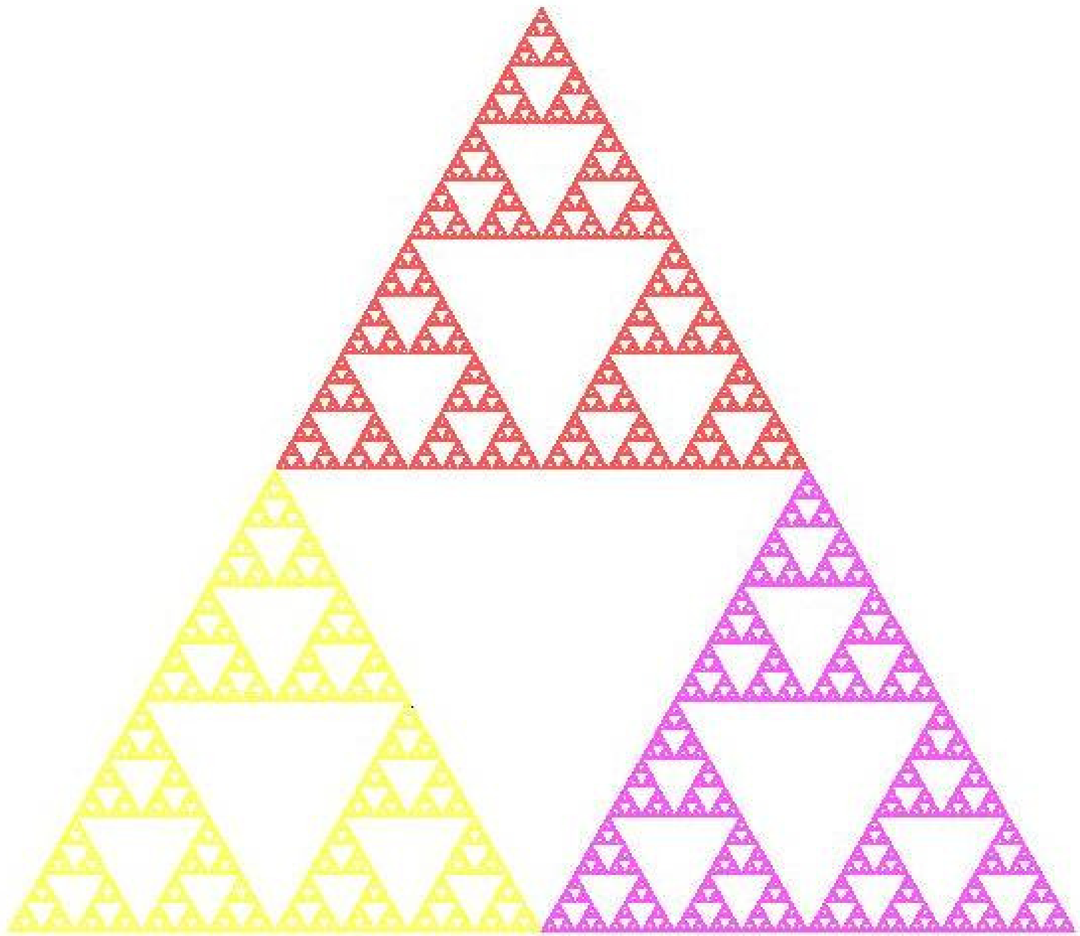
Family $\mathfrak{S} = \{S_1, S_2, S_3\}$ of contractions on $\mathbb{R}^2$, where

$S_1(x) = \frac{1}{2}(x - A) + A, S_2(x) = \frac{1}{2}(x - B) + B, S_3(x) = \frac{1}{2}(x - C) + C$

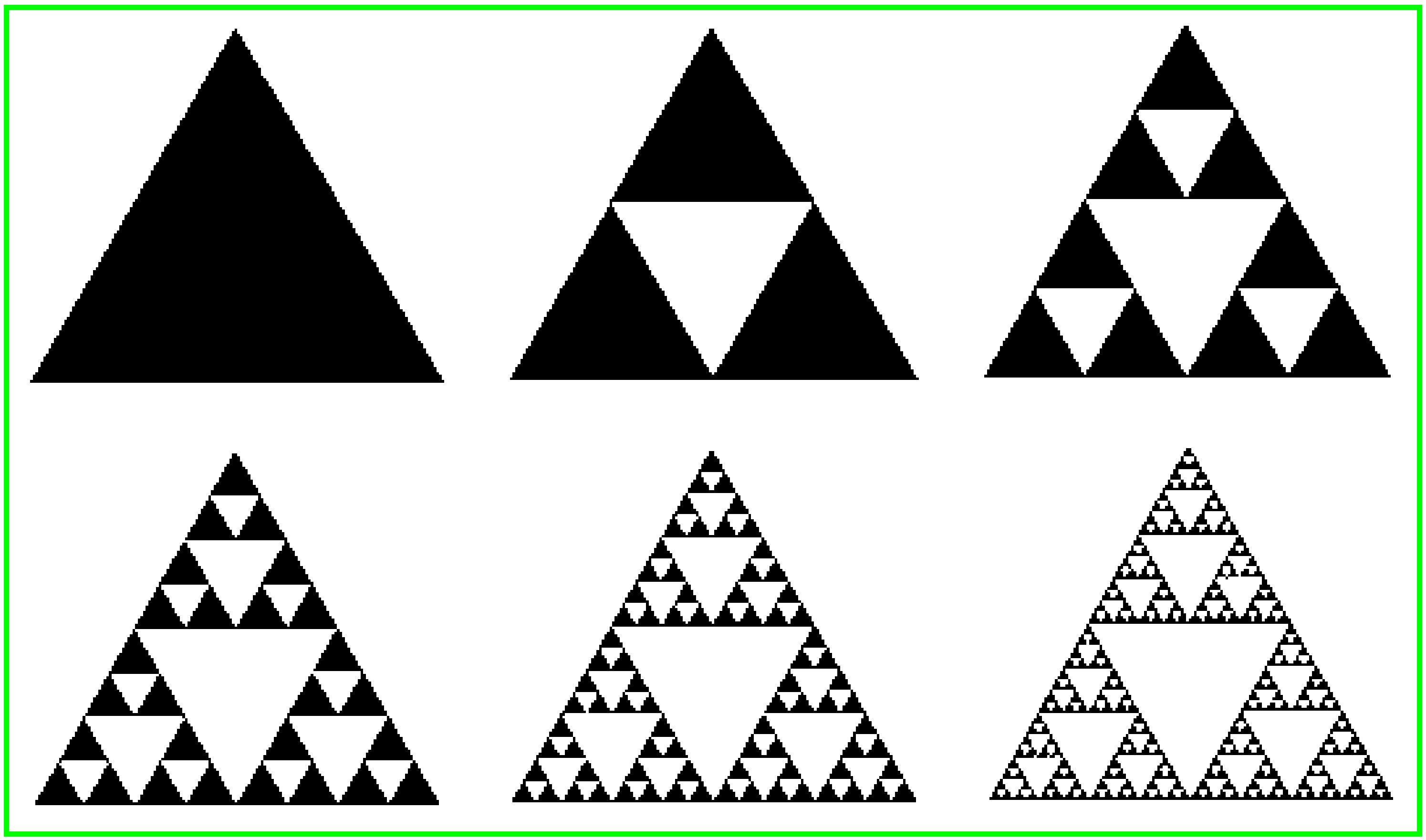
There is a unique (non empty and compact) set K, the so–called Sierpinski gasket:



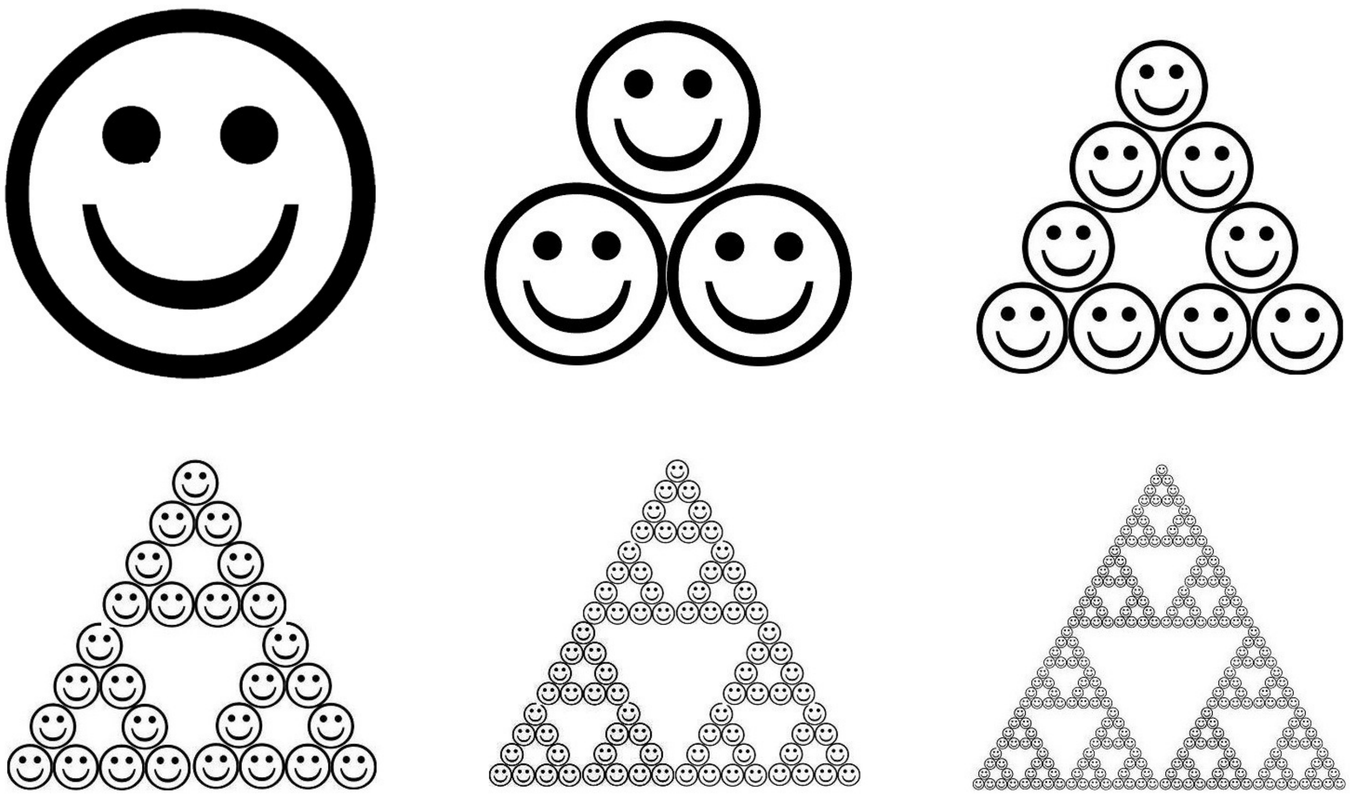
Again:



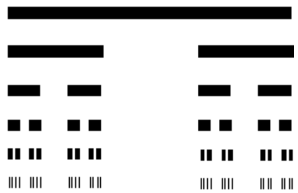
It can be obtained by iteration of the three mappings:



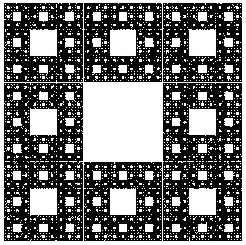
Hereby, you can start with any set:



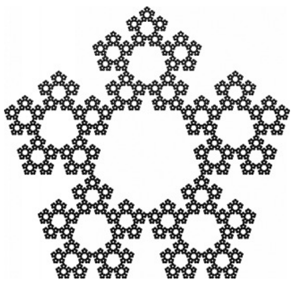
Further examples for self–similarity:



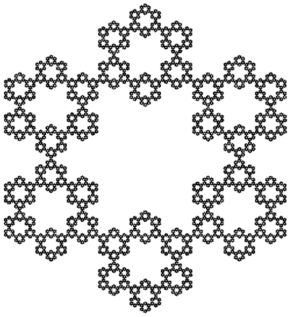
a) Cantor set



b) Sierpinski carpet



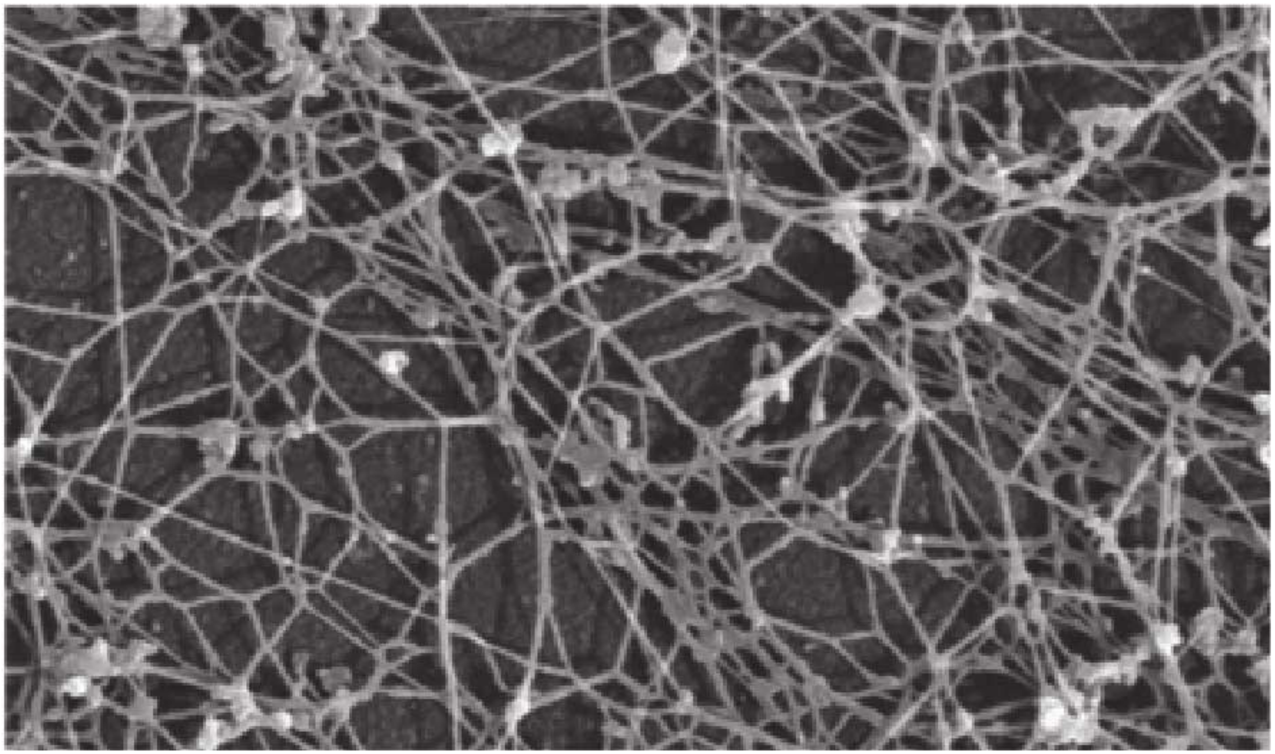
c) Pentagasket



d) Snowflake

1.2. What can we model with the help of fractals?

Application in medicine pulmonary tissue,
tumor cells (and their behavior under stress)



electron microscope picture of the zytoskeleton in a tumor cell in human pancreas

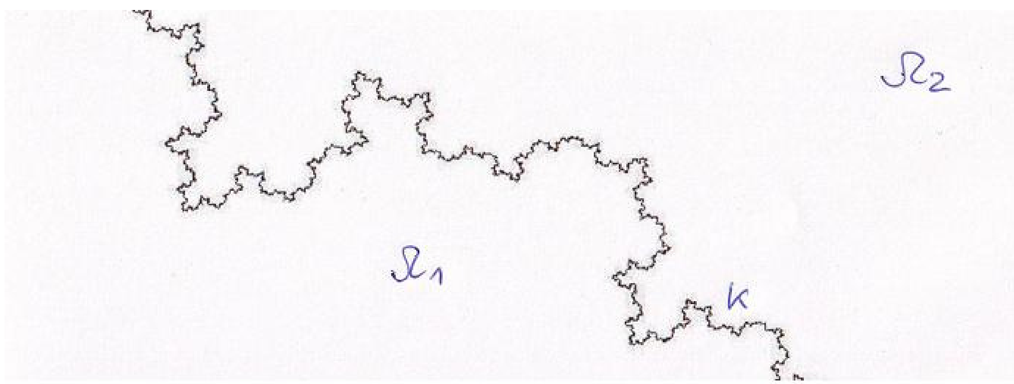
Application in Physics, Material science

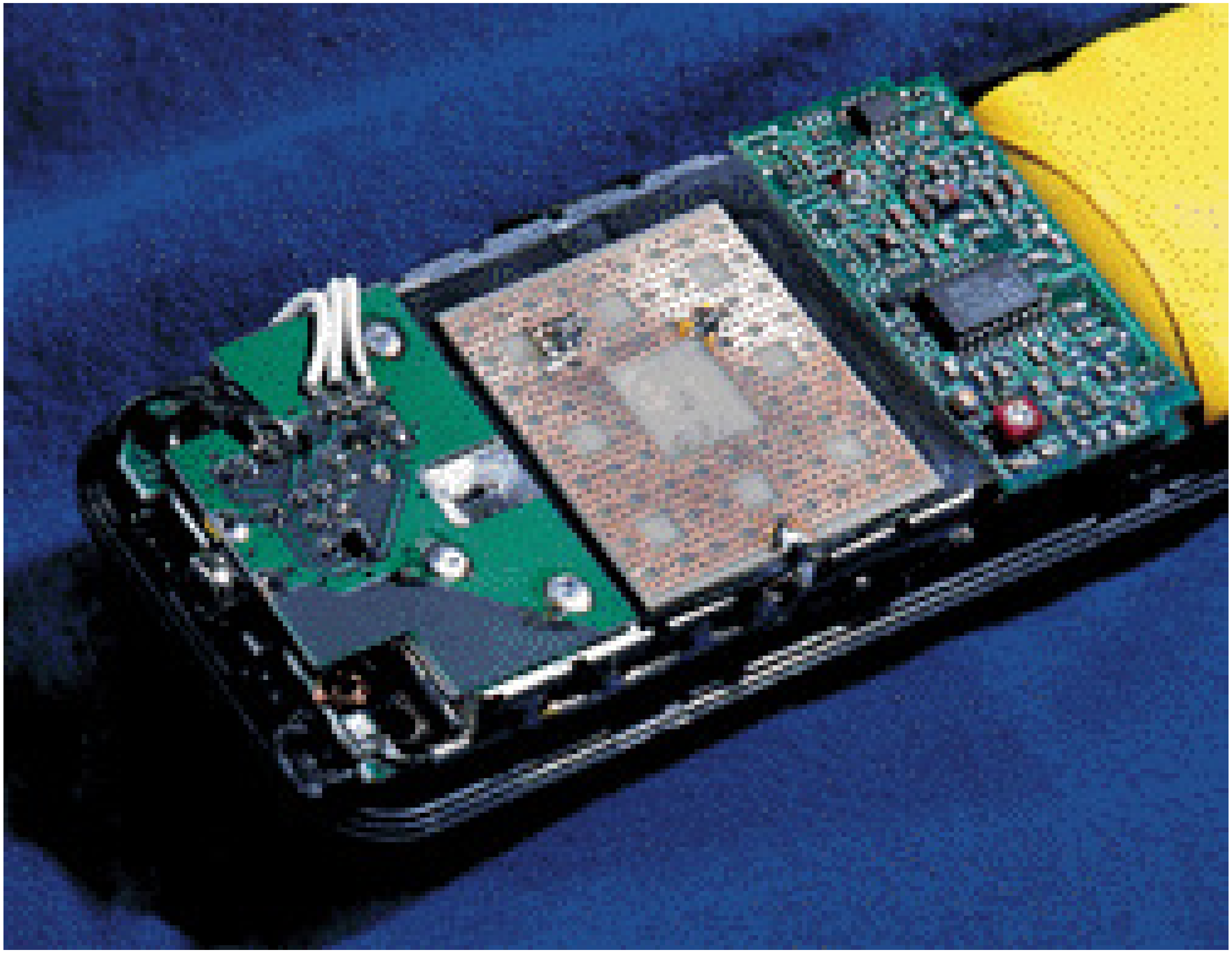
fractal antenna, fractal conductor plates; porous materials

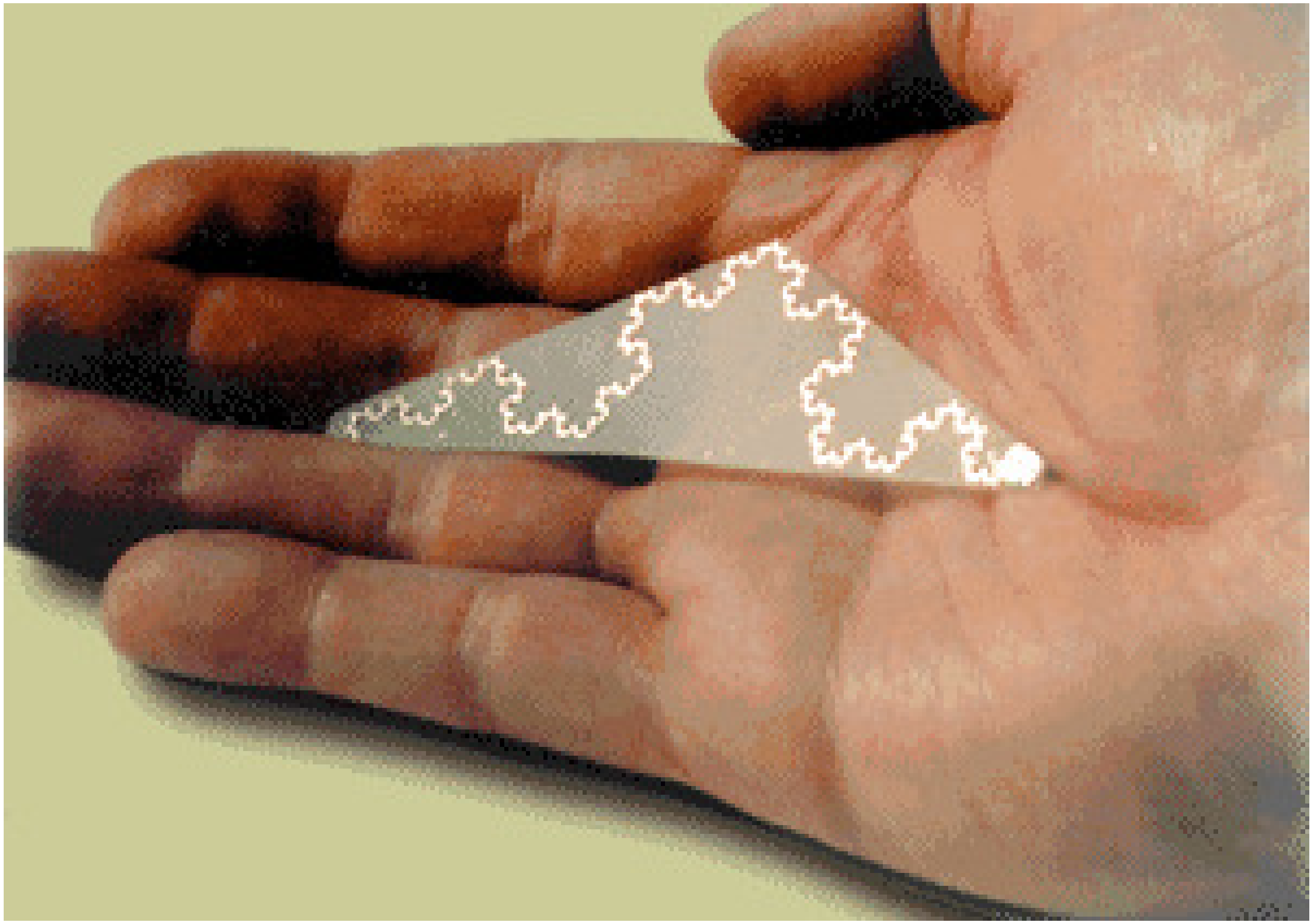
- Transport *on* such structures
- „Transmission problems *across* fractal layers“

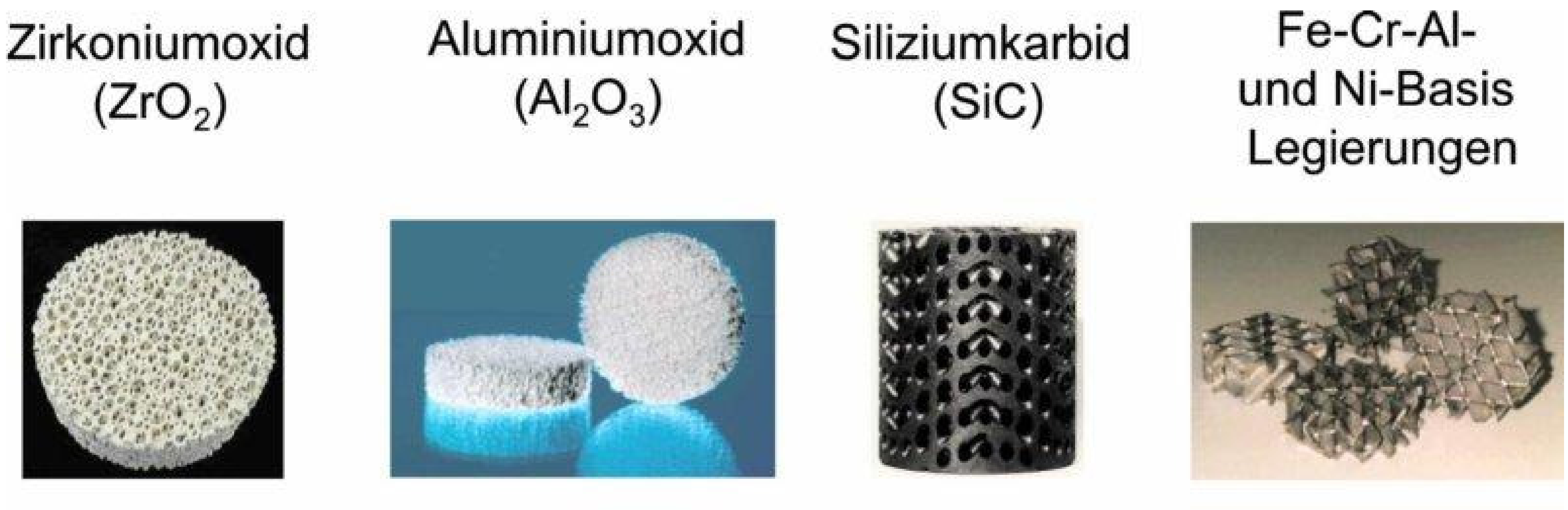
$$\begin{cases} -\Delta_{\mathbb{R}^n} u = g & \text{in } \Omega_i, i = 1, 2 \\ \Delta_K u = C\left(\frac{\partial u_1}{\partial n_1} + \frac{\partial u_2}{\partial n_2}\right) & \text{on } K \end{cases}$$

& boundary and continuity conditions









Porous materials

1.3. Analysis on Fractals

in particular: Definition of a Laplacian Δ
(wave–, heat–, Schrödinger–equation)

Problem: Fractals are to „rough“ „broken“ (non smooth)

$\Longrightarrow$ no notion of tangent space available

$\Longrightarrow$ new approaches necessary

Classical approaches:

- limit of difference operators (Dirichlet form theory)
  Kusuoka, Kigami, Lapidus, Mosco, Hambly, Teplyaev, Strichartz,...

- Construction of the „natural“ Brownian motion as the limit of a sequence of appropriate renormalized random walks
  Kusuoka, Barlow, Bass, Perkins, Lindstrøm; Sabot, Metz,...

- Martin boundary theory on the Code space
  Denker, Sato, Koch,...

- (fractal dimensional) traces of function spaces (for exp. Sobolev spaces) or via Riesz potentials
  Triebel, Haroske, Schmeißer,...; Zähle

New approaches:

- Generalized Laplacians (Δ–Beltrami, Hodge–Δ, Dirac–Δ)
  M. Hinz, Teplyaev, Rogers,...

- Non-commutative Geometry: Interpretation of the fractal in terms of spectral triple
  Bellissard, Falconer, Samuel, Lapidus; Cipriani, Guido, Isola, ...

- Theory of resistance forms
  Kigami, Kajino, Alonso–Ruiz, F. ,...

- Approximation by quantum graphs
  Teplyaev, Kelleher, Alonso–Ruiz, F. ...; Mugnolo, Lenz, Keller, Post, Kuchment, ...

2. Einstein's Relation

$$\frac{d_H}{d_S} = \frac{d_W}{2},$$

where:

d_H Hausdorff dimension $\longleftrightarrow$ geometry

d_S spectral dimension $\longleftrightarrow$ analysis

d_W walk dimension $\longleftrightarrow$ stochastics

Warming up:
Einstein's relation for domains $\Omega \subseteq \mathbb{R}^n$

$\Omega \subseteq \mathbb{R}^n$ open and bounded with smooth boundary $\partial\Omega$

d_H Hausdorff dimension
For open domains $\Omega \subseteq \mathbb{R}^n$ we have $d_H(\Omega) = d_{top}(\Omega)$.
Hence, $d_H(\Omega) = n$.

d_S spectral dimension

of a set is the double of the leading exponent in the asymptotic eigenvalue counting function of its „natural“ Laplacian.

Consider a Dirichlet eigenvalue problem

$$\begin{cases} -\Delta_n u &= \lambda u \text{ on } \Omega \\ u_{|\partial\Omega} &\equiv 0, \end{cases}$$

where $\Delta_n = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the usual Laplacian in $\mathbb{R}^n$.

H. Weyl, 1915: The eigenvalue counting function

$$N_n(x) := \#\{\lambda_k \leq x \quad : \quad -\Delta_n u = \lambda_k u \text{ for some } u \neq 0\},$$

(counting according multiplicities) is well defined, and for any $n \in \mathbb{N}$ it holds that

$$N_n(x) = (2\pi)^{-n} c_n \text{vol}^n(\Omega) x^{n/2} + o(x^{n/2}), \qquad \text{as} \quad x \to \infty,$$

where $\text{vol}^n(\Omega)$ is the n–dimensional volume of Ω and c_n the n–dimensional volume of the unit ball in $\mathbb{R}^n$.
Hence, $d_S(\Omega) = n$.

d_W walk dimension of a set is given by

$$d_w = \frac{\ln \mathbb{E}^x \tau(B(x,R))}{\ln R}, \qquad \text{(i.e. } \mathbb{E}^x \tau(B(x,R)) = R^{d_W}\text{)}$$

where

- $(X_t)_{t \geq 0}$ „natural“ Brownian motion on this set,
- $\tau(B(x,R)) := \inf\{t \geq 0 : X_t \in \partial B(x,R)\}$ and
- $\mathbb{E}^x$ expectation of a random variable if we start in x.

  It is well known that: $d_W(\Omega) = 2$.

  Therefore, $\frac{d_H}{d_S} = \frac{d_W}{2}$ holds, because of $d_H = d_S = n, d_W = 2$.

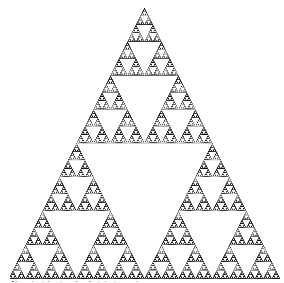
3. Einstein's relation on the Sierpinski gasket

3.1. The geometry of K: the Hausdorff dimension

What kind of geometrical scaling property a „reasonable“ notion of dimension d should provide?

$$\boxed{\text{volume scaling} = \text{length scaling}^{\,d}}$$

The Hausdorff dimension d_H has this property!



Sierpinski gasket $K = \bigcup_{i=1}^{3} \psi_i(K)$

$A := (0,0)$, $B := (1,0)$, $C := (\frac{1}{2}, \frac{\sqrt{3}}{2})$

$\Psi := \{\psi_1, \psi_2, \psi_3\}$, where $\psi_i : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are the unique contractive similitudes with ratio $\frac{1}{2}$ and fixed points A, B and C, respectively.

$d_H(K) = \frac{\ln 3}{\ln 2}$

3.2. Analysis on K: the spectral dimension

Aim: Define Laplacian Δ_K on K

Steps:

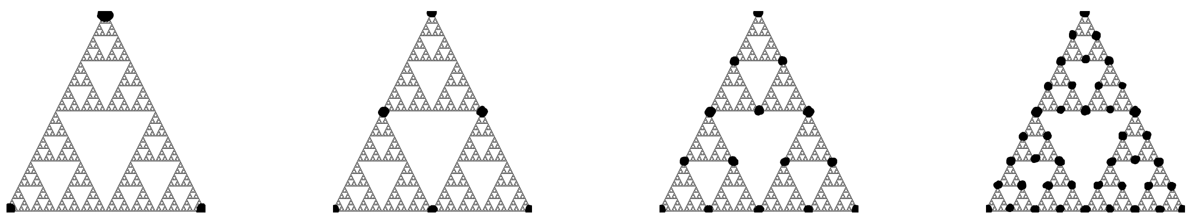
- Define „fractal analogue" $\mathcal{E}_K[u]$ of $\mathcal{E}[u] = \int\limits_{\Omega} |\nabla u|^2 dx$
- $\mathcal{E}_K(u,v) := \frac{1}{2}(\mathcal{E}_K[u+v] - \mathcal{E}_K[u] - \mathcal{E}_K[v])$ bilinear form
- Δ_K via Gauß–Green–formula:

$$\int\limits_K (\Delta_K u)\, v d\mu = \text{boundary terms } - \mathcal{E}_K(u,v)$$

(cf. $\int\limits_{\Omega} \Delta u \cdot v = \text{boundary terms } - \int\limits_{\Omega} \nabla u \cdot \nabla v$)

Approximation of K:

$$V_0 := \{A, B, C\}, \qquad V_n := \bigcup_{i=1}^{3} \psi_i(V_{n-1}), n \geq 1$$



V_0, V_1, V_2 and V_3

$$(V_n) \uparrow, \quad V_* := \bigcup_{n \geq 0} V_n = \sup_{n \geq 0} V_n, \quad K = \overline{V_*}$$

Let be $u : V_* \longrightarrow \mathbb{R}$

Ansatz: $\mathsf{E}_n[u] := \varrho^n \sum_{p \in V_n} \sum_{|p-q|=2^{-n}} (u(p) - u(q))^2, \qquad n \geq 0$

ϱ energy scaling factor (to be determined later)

Let us be given the values of a function u in the three vertices (ergo on the set V_0): $u(A) = u_A$, $u(B) = u_B$ and $u(C) = u_C$.

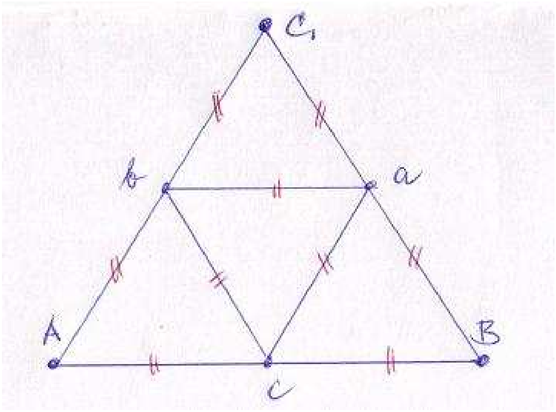
$$\mathcal{E}_0[u] = (u_A - u_B)^2 + (u_A - u_C)^2 + (u_B - u_C)^2$$

ϱ scaling factor is determined from the balance equation

$$\min\{\mathcal{E}_1[v] \mid v : V_1 \longrightarrow \mathbb{R},\ v_{|V_0} = u\} \stackrel{!}{=} \mathcal{E}_0[u] \qquad (1)$$

Hence, seek for the „harmonic extension“ $\tilde{u}$ of u

$$\mathcal{E}_1[u] = \varrho \Big[(u(a) - u_B)^2 + (u(a) - u_C)^2 + (u_B - u(c))^2$$
$$+ (u_A - u(b))^2 + (u_A - u(c))^2 + (u(b) - u_C)^2$$
$$+ (u(a) - u(c))^2 + (u(a) - u(b))^2 + (u(b) - u(c))^2 \Big] \longrightarrow \text{min}$$



$\tilde{u}(a) = (u_A + 2u_B + 2u_C)/5$, $\tilde{u}(b), \tilde{u}(c)$ analogous. Inserting in (1) yields $\varrho = 5/3$.

Self similarity and finite ramification $\Longrightarrow$

$$\min\{\mathcal{E}_n[v] \,\Big|\, v : V_n \longrightarrow \mathbb{R},\ v_{|V_0} = u\} = \mathcal{E}_0[u], \qquad \forall n \geq 1.$$

$\Longrightarrow (\mathcal{E}_n[u])_{n\geq 0}$ non decreasing

defines limit form

$$\mathcal{E}_K[u] := \lim_{n\to\infty} \mathcal{E}_n[u]$$

on

$$\mathcal{D}_* := \{u : V_* \longrightarrow \mathbb{R}\ :\ \mathcal{E}_K[u] < \infty\}$$

Extension of $u \in \mathcal{D}_*$ to $u \in \mathcal{C}(K)$

$\mathcal{D} := \overline{\mathcal{D}_*}$ completion wrt. $\left(||.||^2_{L_2(K,\mu)} + \mathcal{E}_K[.]\right)^{1/2}$

$(\mathcal{E}, \mathcal{D})$ is a Dirichlet form on $L_2(K, \mu)$

$$\int_K (\Delta_K u)\, v d\mu = -\mathcal{E}_K(u, v)$$

Δ_K (Neumann–)Laplacian

Kigami Lapidus, 1993: Spectral dimension of a so–called „nested fractal“ is given by

$$d_S = \frac{2 \ln M}{\ln(M\varrho)}$$

M − number of mappings $S_1, \ldots, S_M$

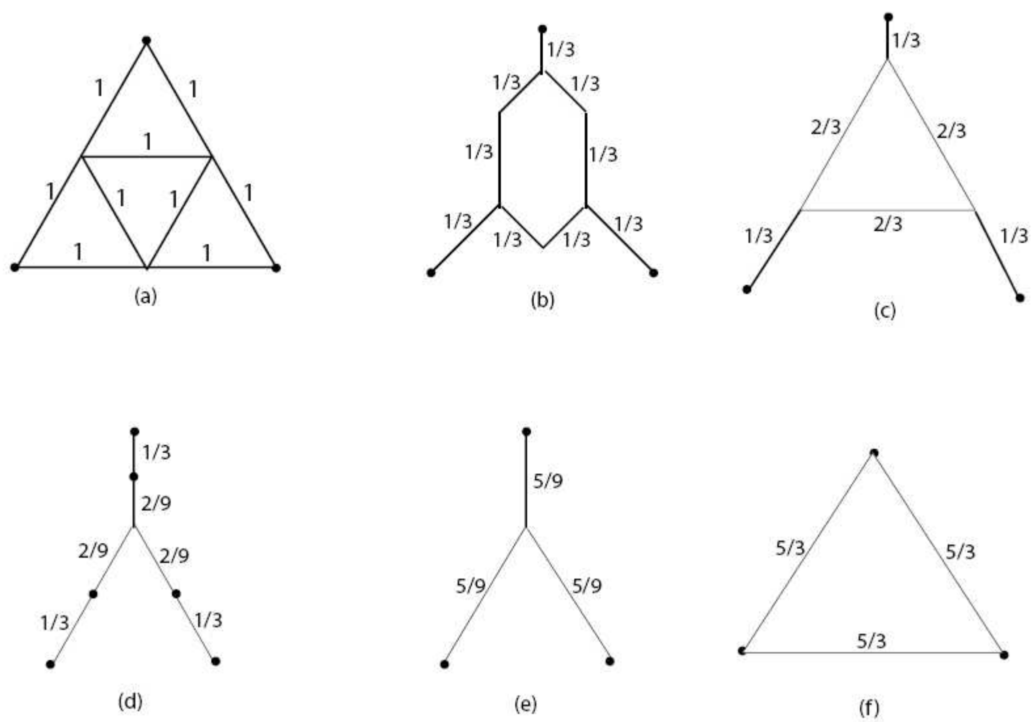
ϱ − energy scaling factor

in the Sierpinski–gasket case: $M = 3, \varrho = 5/3$

$d_S(K) = \frac{\ln 9}{\ln 5}$

Remark on how to find ϱ

Technique: Kirchhoff's law (for ex. „Δ–Y–law“)
($\nearrow$ Graph theory, Analysis on graphs)



Further remarks:

- **Berry's conjecture, early 80's:** We have a Weyl-asymptotics analogue for fractals K, i.e.

$$N_K(x) = c_d \mathcal{H}^d(K) x^{d/2} + o(x^{d/2}), \qquad \text{für} \quad x \to \infty,$$

  where K is a fractal with Hausdorff dimension $d := dim_{\mathsf{H}}(K)$, $\mathcal{H}^d$ is the d–dimensional Hausdorff measure, and c_d is a constant not depending on K. FAILS!
  i.g. $d_H \neq d_S$

- In general it does not hold that: $\mathcal{E}[u] \preceq \mathcal{H}^d$, i.e. we don't have $\mathcal{E}[u] = \int |\nabla u|^2 d\mathcal{H}^d$

- First derivatives are harder to define than second derivatives.

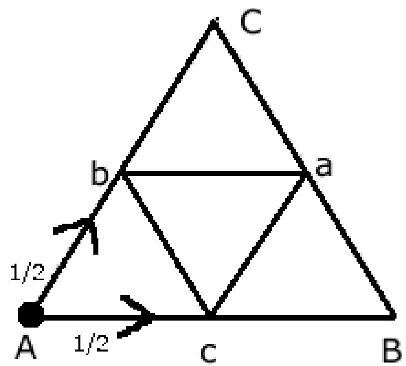
3.3. K as a state space of a BB: Walk–Dimension

$$d_W := \frac{\ln \mathbb{E}^x \tau(B(x,R))}{\ln R}.$$

(actually, in graph theory: $\lim_{R\to\infty}$ of the r.h.s. is taken)

Start in A, calculate the mean random time until we reach B or C.

τ – (random) time of reaching $\{B, C\}$.



Random walk on the graph with vertex set V_1

$$
\begin{aligned}
\mathbb{E}^A\tau &= \frac{1}{2}\Big(\mathbb{E}^b\tau + \mathbb{E}^c\tau\Big) + 1 = \mathbb{E}^c\tau + 1 \\
\mathbb{E}^c\tau &= \frac{1}{4}\Big(\mathbb{E}^A\tau + \mathbb{E}^b\tau + \mathbb{E}^a\tau + \mathbb{E}^B\tau\Big) + 1 = \frac{1}{4}\Big(\mathbb{E}^A\tau + \mathbb{E}^c\tau + \mathbb{E}^a\tau\Big) + 1 \\
\mathbb{E}^a\tau &= \frac{1}{4}\Big(\mathbb{E}^C\tau + \mathbb{E}^b\tau + \mathbb{E}^c\tau + \mathbb{E}^B\tau\Big) + 1 = \frac{1}{2}\mathbb{E}^c\tau + 1
\end{aligned}
$$

Is LES in $(\mathbb{E}^A\tau, \mathbb{E}^c\tau, \mathbb{E}^a\tau)^T$.

Has a unique solution $\mathbb{E}\tau^A = 5$.

$d_W(K) = \frac{\ln 5}{\ln 2}$

„sub–diffusive“

$B^K(t) \stackrel{\mathcal{D}}{=} \alpha^2 B^K(\frac{t}{\alpha^5})$

(with Christoph Thäle, Fribourg, CH) Getting expected crossing times from (only) the connection matrix of the graph

4. Upshot:

So, for the Sierpinski gasket we got $d_H = \ln 3/\ln 2$, $d_S = \ln 9/\ln 5$ and $d_W = \ln 5/\ln 2$.

Obviously, $\frac{d_H}{d_S} = \frac{d_W}{2}$ holds.

„Interpretation“: If you are going to investigate a (porous) set with the EYE (leading to d_H), the EAR (leading to d_S), or the „BLIND–AN–DEAF–ANT“–SENSE (leading to d_W), then it is sufficient to run two of these three experiments.

More general: Take a self–similar nested fractal, then we have:

- $d_H = \frac{\ln M}{\ln L}$ [Hut'81]

- $d_S = 2\frac{\ln M}{\ln(M\varrho)}$ [KigLap'93]

- $d_W = \frac{\ln T}{\ln L}$

where M, L, ϱ, T are mass/length/energy/time scaling numbers.

So, (ER) is equivalent with $T = \varrho M$, i.e.

time= resistance $\times$ mass

Literature

• URF: Einstein relation on fractal objects. Discrete Cont. Dyn. Syst. Ser. B 17 (2012), no. 2, 509–525.

Related/pre works:

• [Telcs'06] The art of Random Walk, Springer
(ER) on graphs, see also [Tetali'91]

$$d_W := \lim_{R\to\infty} \frac{\ln \mathbb{E}^x \tau_R}{\ln R}$$

• [HamKigKum'02] multifractal version of (ER)

$$d_W := \lim_{r\searrow 0} \frac{\ln \mathbb{E}^x \tau_r}{\ln r}$$

is equivalent for self–similar fractals!

- HKE-community: [Grig'21], [Barlow'98], ...

$$p_t(x,y) \sim \frac{c}{t^{\alpha/\beta}} \exp\left(-C \ \frac{d^{\beta(x,y)}}{t}\right)^{\frac{1}{\beta-1}}$$

where $\alpha = \dim_H, \beta = \dim_W$, and

$2 \leq \beta \leq \alpha + 1$ [Barlow'04]

btw: $\beta = \alpha + 1$ for Vicsek

next aims:

find examples st. (ER) fails (with $c = 2$)

study MMS's, stability of (ER) (i.e. of c) under mappings on MMS's
(violate these assumptions in oder so construct non–examples)

• Fabian Burghart, URF: The Einstein Relation on Metric Measure Spaces. (2019, arXiv)
$X =$ Brownian path
$d_H = 3/2$, $d_S = 1$, $d_W = 4$, so we have

$$\frac{d_H}{d_S} = \frac{d_W}{8/3}!$$

5. ER on MMS's

(joint work with F. Burghart, Uppsala; arXiv)

setting: (X, d_X, μ_X) MMS st.

- (X, d_X) Polish, locally cpt., path connected, $\sharp X \geq 2$
- μ_X Radon, supp$\mu_X = X$

In the paper, there are three main parts:

- 5.1 Well–posedness

$\dim_H X$ clear;

conditions for well–definedness of $\dim_S X$ of some operator A acting on $L_2(X, \mu_X)$

conditions for existence of Hunt process $(X_t)_{t\geq 0}$ with i.g. A; $\dim_W$

- 5.2 What gets preserved?

find morphisms $\varphi : (X, d_X, \mu_X) \to (Y, d_Y, \mu_Y)$ st.

(ER; c) invariant; or - even stronger - $\dim_{H,S,W}$ get preserved

- 5.3 find examples st. ER holds with $c \neq 2$

@5.2. $\varphi : (X, d_X, \mu_X) \to (Y, d_Y, \mu_Y)$

• φ bi–Lipschitz + measure–preserving $\Rightarrow$ $\dim_H, \dim_W$ invariant

(φ homeomorphic + measure–preserving $\Rightarrow$ $\dim_S$ invariant)

So, φ bi–Lipschitz + measure–preserving $\Rightarrow$ $(ER; c)$ preserved

• φ homeomorphic, α–Hölder, $+$ measure–preserving $\Rightarrow$ $\dim_S$ preserved

$\dim_H \varphi(X) \leq \frac{1}{\alpha} \dim_H X$ and

$\overline{\dim}_W(\varphi(X), \varphi(M), \varphi(x)) \leq \frac{1}{\alpha} \dim_W(X, M, x)$

$M = (M_t)_t$ Hunt process on X

$\overline{\dim}_W(X, M, x) := \overline{\lim}_{r \searrow 0} \frac{\log \mathbb{E}^x \tau_{M,B(x,r)}}{\log r}$

special cases: graphs of α–Hölder functions

@5.3. Counter–Examples

$B^H = (B_t^H)_t$ fractional BM with Hurst index $H \in (0,1)$

$X :=$ graphB^H

- $\dim_H X = 2 - H$ a.s. (Adler, 77)
- $\dim_S X = 1$ a.s.
- $\dim_W X = 2/H$ a.s.

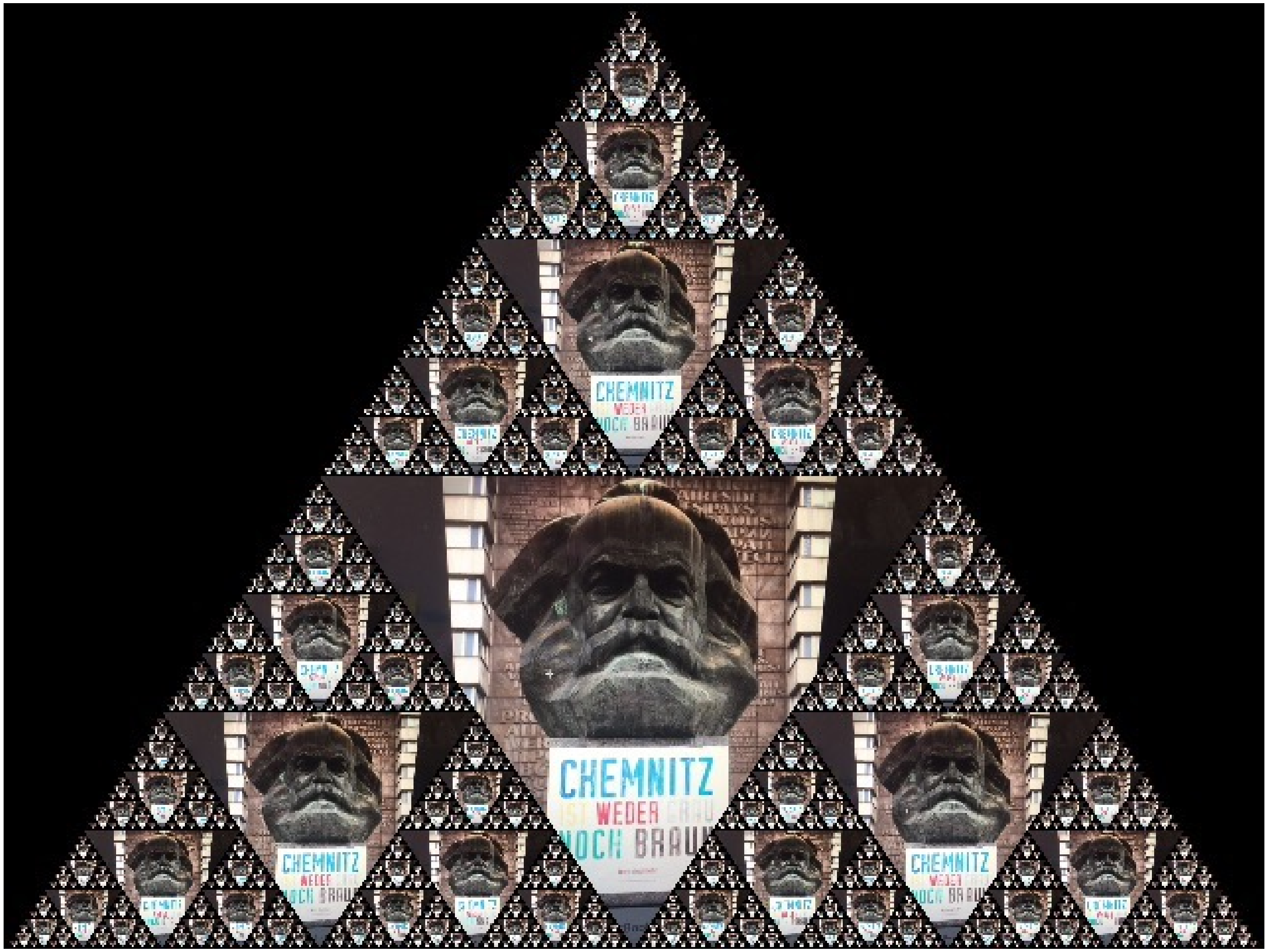
so, $c = \frac{2}{(2-H)H}$

$c \in (2, \infty)$!

Open Problems/Remarks

- counter examples are „comb–like“ so far
- lower estimate for $\underline{\dim}_W$, *time distribution principle* ??
- *effective* $\dim_H$?? ($< \dim_H$)
- for fixed c: minimize/maximize $\dim_{H,S,W}$
- ...

Thank you for your attention!



Save the date: FGS7 in Chemnitz in autumn 2024!